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LETTER TO THE EDITOR

One-dimensional Ising spin-glass model with long-range interactions

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Abstract. An Ising chain with Hamiltonian $H=-J\sum_{i<j}\varepsilon_{ij}S_iS_j/|i-j|^{(1+\sigma)/2}$ is considered where the ε_{ij} are independent random variables. According to Kotliar *et al* this model has a phase transition with non-mean-field exponents when $\frac{1}{3}<\sigma<1$ and mean-field exponents beyond the upper critical value of $\sigma=\frac{1}{3}$. By means of an ε expansion about the lower critical value of $\sigma=1$, the model is shown to be replica symmetric as $\varepsilon\to 0$ and it is speculated that this result holds for $\frac{1}{3}<\sigma<1$.

The rationale for the study of one-dimensional spin models with long-range interactions is that they may serve as instructive analogies for higher-dimensional models with short-range interactions. To this end Kotliar *et al* (1983, to be referred to hereafter as KAS) introduced a model one-dimensional spin-glass Hamiltonian

$$H\{S_i\}=-\sum_{\langle ij\rangle}J_{ij}S_iS_j \tag{1}$$

where the Ising spins S_i take the values ±1, the sum is over all pairs $\langle ij\rangle$ and i and j denote integer positions on a one-dimensional lattice. The interaction

$$J_{ij}=J\frac{\varepsilon_{ij}}{|i-j|^{(1+\sigma)/2}} \tag{2}$$

where the ε_{ij} are independent random variables with a Gaussian distribution of zero mean and unit variance. This model is clearly highly artificial and no physical system comes close to being well described by it. Nevertheless, there are useful similarities between this model's properties as a function of the range parameter σ and the behaviour of short-range models as a function of their dimensionality d. For example KAS demonstrate that for $-1\leqslant\sigma<\frac{1}{3}$ the one-dimensional model has mean-field behaviour and exponents. Indeed, provided J is rescaled to ensure the existence of the thermodynamic limit, the model reduces for $\sigma=-1$ to the much studied Sherrington-Kirkpatrick (SK) model (Sherrington and Kirkpatrick 1975). For short-range spin glasses, mean-field exponents are expected for all dimensions from infinity down to six. Between six and the lower critical dimension (whose precise value is not known but probably lies between two and three dimensions (see Bray and Moore (1984, 1985) for recent speculations)) non-mean-field exponents are obtained and this is parallelled by non-mean-field exponents in the long-range model for $\frac{1}{3}<\sigma<1$. For $\sigma>1$ the model becomes effectively short-ranged and does not exhibit a transition at finite temperatures.

The nature of the low temperature phase in spin glasses has only been determined for the SK model. The phase is characterised by many pure states (Parisi 1983) whose free energies are close to the one of lowest free energy and the space of pure states displays an 'ultrametric' topology (Mézard *et al* 1984a, b). Certain quantities such as the susceptibility are not self-averaging in the thermodynamic limit (Young *et al* 1984). Strong correlations exist between the magnetisations $m_i^s(=\langle S_i\rangle$ in state S) of different pure states. These can be described by the Parisi overlap functions (Parisi 1983, Mézard *et al* 1984a, b), the simplest of which is

$$P_J(q)=\sum_{S,S'} P_s P_{s'}\delta(q-q_{ss'}). \tag{3}$$

P_s is the Boltzmann weight of the pure state S and $q_{ss'}$ is the overlap $(1/N)\sum_i m_i^s m_i^{s'}$. The configuration averaged $\overline{P_J(q)}=P(q)$ is non-trivial as replica symmetry breaking occurs in the SK limit.

The natural assumption to make about the ordered phase outside the SK model (which corresponds to the limit of infinite dimensionality) is that it is similar to that of the SK model in having many pure states. However, we have argued elsewhere (Moore and Bray 1985) that for $d<6$ there is only a single pure state (or rather two pure states, counting the time-reversed state). This implies a trivial $P(q)$:

$$P(q)=\tfrac{1}{2}[\delta(q-q_{EA})+\delta(q+q_{EA})]$$

where q_{EA} is the Edwards–Anderson order parameter (Edwards and Anderson 1975). In the language of replicas the low temperature phase is replica symmetric. We hypothesised that replica symmetry pertains for $d<6$, but that for $d>6$ many pure states and replica symmetry breaking should exist. In the mean-field limit there is a line in the field-temperature diagram—the so-called Almeida-Thouless (AT) line (de Almeida and Thouless 1978)—which separates the high temperature paramagnetic phase from a phase characterised by many pure states. We argued that as for $d<6$ the ordered phase is replica symmetric, there should be no AT line for $d<6$.

All the arguments which we advanced for these assertions should apply with equal force to the long-range one-dimensional model when $\frac{1}{3}<\sigma<1$, i.e. in the region of non-mean-field exponents. On the basis of these arguments we expect a trivial replica symmetric $P(q)$ for all values of σ between the upper critical value of $\frac{1}{3}$ and the lower critical value of unity. The chief result of this letter is that $P(q)$ is indeed trivial near the lower critical value of σ, but first I shall indicate how just one of our arguments (Moore and Bray 1985) for a trivial $P(q)$ extends to the long-range model. Bray and Roberts (1980) performed an ε expansion for the exponents across the presumed AT line in $(6-\varepsilon)$ dimensions, but were unable to find a stable accessible fixed point. One possible explanation of this could be that there is simply no AT line in $(6-\varepsilon)$ dimensions. A similar expansion about the upper critical value of $\sigma(=\frac{1}{3})$ can be done for the one-dimensional long-range spin glass. One has simply to set in the Bray–Roberts calculation the exponent η equal to $(2-\sigma)$—the value appropriate to this long-range interaction (Sak 1973)—and $\varepsilon=5$. The effective expansion parameter is $(3\sigma-1)/2$. Once more no stable accessible perturbative fixed point can be located, suggesting that for the long-range model there is no AT line when $\frac{1}{3}<\sigma<1$.

To investigate the system near the lower critical value of σ, i.e. around $\sigma=1$, we follow the procedure of KAS. They first used the replica method to write the average of the nth power of the partition function Z as

$$\overline{Z^n}=\operatorname*{Tr}_{\{S_i^\alpha\}}\int\prod_{\langle ij\rangle}\mathrm{d}\varepsilon_{ij}P(\varepsilon_{ij})\exp\Big(\beta J\sum_{i<j}\varepsilon_{ij}\sum_\alpha S_i^\alpha S_j^\alpha/|i-j|^{(1+\sigma)/2}\Big). \tag{4}$$

Carrying out the ε_{ij} integrals gives

$$\overline{Z^n} = \operatorname{Tr}\exp(-\beta H_n) = \operatorname*{Tr}_{\{S_i^\alpha\}} \exp\left(\tfrac{1}{2}\beta^2 J^2 \sum_{i<j}\sum_{\alpha,\beta} S_i^\alpha S_i^\beta S_j^\alpha S_j^\beta / |i-j|^{(1+\sigma)}\right). \tag{5}$$

The free energy

$$F = -(\beta n)^{-1} \ln \overline{Z^n} \qquad \text{as } n\to 0. \tag{6}$$

At each site i there are replicated spin variables $S_i^\alpha = \pm 1$, $\alpha = 1, 2, \ldots, n$. The possible values of S_i^α can be regarded as lying at the vertices of an n-dimensional hypercube. KAS found it convenient to introduce variables $\boldsymbol{\sigma}_{\alpha i}$, $\alpha = 1, \ldots, 2^n$ to describe the 2^n different values of $\{S_i^\alpha\}$. The $\boldsymbol{\sigma}_\alpha$ are n-component vectors, e.g.

$$\boldsymbol{\sigma}_1 = (1, 1, \ldots, 1) \qquad \boldsymbol{\sigma}_2 = (-1, 1, \ldots, 1) \qquad \boldsymbol{\sigma}_3 = (1, -1, \ldots, 1) \qquad \text{etc.} \tag{7}$$

The next step is to write the Hamiltonian H_n of equation (5) in terms of defect or 'kink' variables and determine the renormalisation group (RG) equations for the defect fugacities and coupling constants (Cardy 1981).

The interaction energy between a spin in state α at site i and a spin in state β at site j is

$$\frac{K(\alpha,\beta)}{|i-j|^{(1+\sigma)}} \equiv \tfrac{1}{2}\beta^2 J^2 \frac{[(\boldsymbol{\sigma}_\alpha \cdot \boldsymbol{\sigma}_\beta)^2 - n^2]}{|i-j|^{(1+\sigma)}} \tag{8}$$

where a constant has been subtracted from H_n in order that $K(\alpha,\alpha) = 0$. Let $Y_{\alpha\beta}$ be the fugacity of an $\alpha\beta$ defect. The partition function $\overline{Z^n}$ is given in kink variables by

$$\begin{aligned}\overline{Z^n} = \sum_{p=0}^{\infty} \sum_{\alpha_1\ldots\alpha_p}^{2^n} & Y_{\alpha_1\alpha_2} Y_{\alpha_2\alpha_3} \ldots Y_{\alpha_p\alpha_1} \int \frac{\mathrm{d}r_1}{a} \ldots \int \frac{\mathrm{d}r_p}{a} \\ & \times \prod_i \theta(r_{i+1} - r_i - a) \exp\left\{ -[\sigma(1-\sigma)]^{-1} \sum_{q<p} \left[\left(\frac{r_p - r_q}{a}\right)^{1-\sigma} - 1\right] \right. \\ & \left. \times [K(\alpha_{p+1}, \alpha_q) + K(\alpha_p, \alpha_{q+1}) - K(\alpha_p, \alpha_q) - K(\alpha_{p+1}, \alpha_{q+1})] \right\}\end{aligned} \tag{9}$$

(Cardy 1981, Kosterlitz 1976, KAS). Note that $Y_{\alpha\alpha} \equiv 0$ and that periodic boundary conditions have been assumed; a is the lattice spacing. A change in the lattice spacing $a \to a \exp l$ can be compensated by a change in the kink fugacities and coupling constants provided the partition function remains invariant. This leads to the RG equations (KAS)

$$\frac{\mathrm{d}Y_{\alpha\beta}}{\mathrm{d}l} = Y_{\alpha\beta}\left(1 + \frac{2}{\sigma} K(\alpha,\beta)\right) + \sum_{\nu\neq\alpha,\beta} Y_{\alpha\nu} Y_{\nu\beta} \tag{10}$$

$$\begin{aligned}\frac{\mathrm{d}K}{\mathrm{d}l}(\alpha,\beta) = (1-\sigma)K(\alpha,\beta) & - \sum_\nu Y_{\alpha\nu}^2 [K(\alpha,\beta) - K(\beta,\nu) + K(\alpha,\nu)] \\ & - \sum_\nu Y_{\beta\nu}^2 [K(\alpha,\beta) - K(\alpha,\nu) + K(\beta,\nu)].\end{aligned} \tag{11}$$

Equations (10) and (11) should be valid to first order in ε and small fugacities where $\varepsilon = 1 - \sigma$. KAS claim that the pertinent fixed point of equations (10) and (11) is the symmetric fixed point

$$Y_{\alpha\beta} = Y^* = (\varepsilon/2^{n+1})^{1/2} = (\varepsilon/2)^{1/2} \qquad \text{as } n \to 0 \tag{12}$$

$$K(\alpha,\beta)=K^*=[-1+(2-2^n)(\varepsilon/2^{n+1})^{1/2}]/2+O(\varepsilon)$$
$$=-\tfrac{1}{2}+(\varepsilon/2)^{1/2}/2 \qquad \text{as } n\to 0. \tag{13}$$

By linearising around this fixed point, KAS showed that the correlation length exponent ν is given by $1/\nu=1.1\sqrt{\varepsilon}$.

We next proceed to calculate the magnetic exponent y_h which describes the dependence of the field conjugate to the spin-glass order parameter (see below). By scaling,

$$y_h=(d+2-\eta)/2. \tag{14}$$

Now for systems with long-range interactions $\eta=2-\sigma$ (Sak 1973) so

$$y_h=(1+\sigma)/2=1-\varepsilon/2. \tag{15}$$

It is instructive to see how this result emerges from RG recursion relations. A conjugate field for spin glasses derives from a Gaussian random field h_i of zero mean and variance h. We generalise the Hamiltonian of equation (1) to

$$H=-\sum_{\langle ij\rangle} J_{ij}S_iS_j-\sum_i h_iS_i.$$

After replicating and averaging over $\{J_i\}$, and $\{h_i\}$, the βH_n term of (5) has to be supplemented by the expression

$$\tfrac{1}{2}\beta^2h^2\sum_i\left(\sum_{\alpha=1}^{n} S_i^\alpha\right)^2. \tag{16}$$

The form of (16) suggests that β^2h^2 is the conjugate field for the quantity

$$\frac{1}{N}\sum_i\sum_{\alpha=1}^{n}\sum_{\beta=1}^{\alpha}\langle S_i^\alpha S_i^\beta\rangle$$

which is related to the 'statistical mechanics' spin-glass order parameter (de Dominicis and Young 1983). Let $\boldsymbol{t}$ denote the n-component vector $(1,1,1,\ldots,1)$. Then (16) can be rewritten in terms of $\boldsymbol{t}$ and the $\boldsymbol{\sigma}_{\alpha i}$ variables as

$$\tfrac{1}{2}\beta^2h^2\sum_i(\boldsymbol{t}\cdot\boldsymbol{\sigma}_{\alpha i})^2=\sum_i H_\alpha(i) \tag{17}$$

where $H_\alpha=\tfrac{1}{2}\beta^2h^2(\boldsymbol{t}\cdot\boldsymbol{\sigma}_\alpha)^2$. RG equations for the H_α will now be derived, assuming that h is infinitesimally small. Due to the new field terms, equation (9) will now contain an additional factor in the integrand of

$$\exp[H_{\alpha_2}(r_2-r_1)/a+H_{\alpha_3}(r_3-r_2)/a+\ \ldots\ +H_{\alpha_1}(L-|r_p-r_1|)/a] \tag{18}$$

where L is the length of the chain. If we replace a by $a\,\mathrm{e}^l$, each H_α in (18) must be changed to $H_\alpha\,\mathrm{e}^l$ in order to leave Z^n unmodified. The effect of changing a in the cut-off is most easily obtained by writing

$$\theta(r_{j+1}-r_j-a\,\mathrm{e}^l)=\theta(r_{j+1}-r_j-a)-al\delta(r_{j+1}-r_j-a)+O(l^2) \tag{19}$$

and neglecting the terms of $O(l^2)$ (Cardy 1981). The delta function means that the additional terms from changing the cut-off can be derived by juxtaposing each neighbouring kink pair in turn. The factors involving two such kinks i, $i+1$ and a third kink at j of the type (α, β) are

$$lY_{\alpha_1\alpha_2}Y_{\alpha_2\alpha_3}[(r_j-r_i)/a]^{(K(\alpha_1,\alpha)+K(\alpha_2,\beta)-K(\alpha_1,\beta)-K(\alpha_2,\alpha))}$$
$$\times[(r_j-r_i-a)/a]^{(K(\alpha_2,\alpha)+K(\alpha_3,\beta)-K(\alpha_2,\beta)-K(\alpha_3,\alpha))}$$
$$\times\exp[H_{\alpha_1}(r_i-r_{i-1})/a+H_{\alpha_2}+H_{\alpha_3}(r_{i+2}-r_i-a)/a]. \tag{20}$$

If $\alpha_1=\alpha_3$ the kink pair at i and $i+1$ is referred to as neutral. For $(r_j-r_i)\gg a$, $(r_j-r_i-a)/a$ can be approximated by $(r_j-r_i)/a$ to leading order, when for a neutral pair, equation (20) becomes independent of r_i. Terms beyond the leading order depending on the fields H_α lead to modifications of $O(H_\alpha)$ in the recursion relations (10) and (11) for $Y_{\alpha\beta}$ and $K(\alpha, \beta)$ and can therefore be dropped for small H_α, as can the field contributions from non-neutral pairs, i.e. those where $\alpha_1\neq\alpha_3$. Integrating r_i from r_{i-1} to r_{i+2} reduces (20) to

$$lY^2_{\alpha_1\alpha_2}[(r_{i+2}-r_{i-1})/a]\exp[H_{\alpha_1}(r_{i+2}-r_{i-1})/a][1+(H_{\alpha_2}-H_{\alpha_1})+O(H^2_\alpha)]. \tag{21}$$

The term of O(1) in (21) contributes only a constant to the free energy (Cardy 1981) but the term of $O(H_{\alpha_2}-H_{\alpha_1})$ has to be compensated by a change in H_{α_1} to $(H_{\alpha_1}+\Delta H_{\alpha_1})$, namely

$$\exp[(H_{\alpha_1}+\Delta H_{\alpha_1})(r_{i+2}-r_{i-1})/a]=\exp[H_{\alpha_1}(r_{i+2}-r_{i-1})/a]$$
$$\times[1+\Delta H_{\alpha_1}(r_{i+2}-r_{i-1})/a+\ldots]. \tag{22}$$

Summing over all possible neutral pairs, the final RG equation for H_α is

$$\frac{dH_\alpha}{dl}=H_\alpha+\sum_{\nu\neq\alpha}Y^2_{\alpha\nu}(H_\nu-H_\alpha). \tag{23}$$

Put $H_\alpha=HC_\alpha$ where initially $C_\alpha=(\boldsymbol{t}\cdot\boldsymbol{\sigma}_\alpha)^2$ and $H=\frac{1}{2}\beta^2h^2$. At the symmetric fixed point $Y_{\alpha\nu}=Y^*$ so (23) reduces there to

$$\frac{dH}{dl}=H[1-2^n(Y^*)^2]=H(1-\varepsilon/2) \qquad \text{as } n\to0 \tag{24}$$

where we have used the result that $\Sigma_\alpha C_\alpha=0$ as $n\to0$. Hence the magnetic exponent y_h calculated from (24) coincides with the scaling prediction of equation (15).

The calculation of $P(q)$ follows along similar lines. Parisi (1983) has shown that a definition of $P_J(q)$ equivalent to that of equation (3) is

$$P_J(q)=\left\langle\delta\left(q-(1/N)\sum_i T_iS_i\right)\right\rangle \tag{25}$$

provided the thermal average is now over the doubly replicated Hamiltonian $H\{T_i\}+H\{S_i\}$. It is useful to introduce

$$F_J(y)\equiv\left\langle\exp\left((y/N)\sum_i T_iS_i\right)\right\rangle \tag{26}$$

for then

$$P_J(q)=\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}e^{-yq}F_J(y). \tag{27}$$

In order to calculate the bond average of $\overline{F_J(y)}(=F(y))$ we again introduce replicated variables at each site, S_i^α, T_i^α, $\alpha=1,\ldots,n$. Now equation (26) means when explicitly written out

$$F_J(y) = \operatorname*{Tr}_{\{T_i\}\{S_i\}} \exp\Big((y/N)\sum_i T_i S_i - \beta H\{S_i\} - \beta H\{T_i\}\Big)$$
$$\times\Big(\operatorname*{Tr}_{\{T_i\}} \exp[-\beta H\{T_i\}] \operatorname*{Tr}_{\{S_i\}} \exp[-\beta H\{S_i\}]\Big)^{-1}. \tag{28}$$

The terms in the denominator can be re-expressed as $n\to 0$ as $(Z^{n-1})^2$ so

$$F_J(y) = \operatorname*{Tr}_{\{T_i^\alpha\}\{S_i^\alpha\}} \exp\Big((y/N)\sum_i T_i^1 S_i^1 - \sum_{\alpha=1}^{n} (\beta H\{S_i^\alpha\} + \beta H\{T_i^\alpha\})\Big). \tag{29}$$

After bond averaging the calculation of $F(y)$ proceeds as previously. The kink or defect variable $\boldsymbol{\sigma}_\alpha$ is now a vector of length $2n$, the first n components contain the S variables, the second n components the T variables. α runs from $1\text{-}2^{2n}$. Because at the end of the calculation n must be set to zero, the analogues of equations (10) and (11) in the new variables $K(\alpha,\beta)$ and $Y_{\alpha\beta}$, $\alpha = 1\text{-}2^{2n}$ yield the same fixed points and exponents. The term in y produces a field term (cf equation (18))

$$\exp[H_{\alpha_2}(r_2 - r_1)/a + H_{\alpha_3}(r_3 - r_2)/a + \ldots + H_{\alpha_1}(L - |r_p - r_1|)/a]$$

where now $H_{\alpha_i} = (y/N) S^1_{\alpha i} T^1_{\alpha i}$; $S^1_{\alpha_i}$ and $T^1_{\alpha_i}$ are the first and $(n+1)$th components of $\boldsymbol{\sigma}_{\alpha i}$. Notice that because of the factor $1/N$, H_{α_i} is very small and so the derivation of the analogue of equation (23), namely

$$\frac{\mathrm{d}H_\alpha}{\mathrm{d}l} = H_\alpha + \sum_{\nu\neq\alpha} Y^2_{\alpha\nu}(H_\nu - H_\alpha) \qquad \alpha = 1,\ldots,2^{2n} \tag{30}$$

is well founded. Putting $H_\alpha = HC_\alpha$ where now initially $C_\alpha = S^1_\alpha T^1_\alpha$ and $H = y/N$, one observes that $\Sigma C_\alpha = 0$. Assuming that $Y_{\alpha\nu} = Y(l)$, at least in the vicinity of the symmetric fixed point, equation (30) reduces to

$$\mathrm{d}H/\mathrm{d}l = H[1 - 2^{2n} Y^2(l)]. \tag{31}$$

At the fixed point itself, where $Y^* = (\varepsilon/2^{2n+1})^{1/2}$, equation (31) becomes

$$\mathrm{d}H/\mathrm{d}l = H(1 - \varepsilon/2) \tag{32}$$

and hence the magnetic exponent y_h obtained this way is identical to that previously obtained by calculating the response to a random field.

As $n\to 0$, we can set 2^{2n} in equation (31) to unity, and solve for $H(l)$, with $H(0) = y/N$;

$$H(l) = (y/N)\exp\Big(l - \int_0^l Y^2(l')\,\mathrm{d}l'\Big). \tag{33}$$

At the value of $l = l^*$ where $\mathrm{e}^{l^*} = N$, there are effectively just two spins S and T left in the system coupled by $H(l^*)$—the rest have been integrated out.

Define

$$q_{\mathrm{EA}} = \exp\Big(-\int_0^{l^*} Y^2(l')\,\mathrm{d}l'\Big) \qquad \text{as } l^*\to\infty. \tag{34}$$

Then (cf equation (28))

$$F(y) = \operatorname*{Tr}_{T,S} \exp(H(l^*)TS)\Big(\operatorname*{Tr}_{T}(1)\operatorname*{Tr}_{S}(1)\Big)^{-1} = (\mathrm{e}^{yq_{\mathrm{EA}}} + \mathrm{e}^{-yq_{\mathrm{EA}}})/2$$

so from equation (27)

$$P(q)=[\delta(q-q_{\mathrm{EA}})+\delta(q+q_{\mathrm{EA}})]/2. \tag{35}$$

Equation (35) shows that at least to first order in ε, $P(q)$ is trivial and that the spin glass is apparently replica symmetric, i.e. it consists of basically a single pure state. It is difficult to see how higher-order terms in the ε expansion could ever modify this result. We suspect, therefore that for $\frac{1}{3}<\sigma<1$ replica symmetry is obtained. For $\sigma<\frac{1}{3}$ no RG treatment is possible; one proceeds via a loop expansion about the mean-field theory. When this is a valid procedure we expect replica symmetry breaking to occur. Bhatt and Young (1985) have performed Monte Carlo simulations for the one-dimensional long-range model, but to date they have not calculated $P(q)$ outside the mean-field limit. A Monte Carlo simulation of $P(q)$ would be a highly desirable check on these calculations.

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